VỀ SỰ TỒN TẠI HAI NGHIỆM KHÔNG TẦM THƯỜNG CHO BÀI TOÁN DIRICHLET CHỨA TOÁN TỬ p-LAPLACE THỨ

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TÓM TẮT

Bài báo này, chúng tôi nghiên cứu sự tồn tại hai nghiệm yếu cho bài toán biên Dirichlet chứa toán tử không địa phương

$$\begin{cases} \mathcal{L}_K^p u &= \gamma f(x,u) \text{ trong } \Omega \\ u &= 0 \quad \text{trong } \mathbb{R}^N \setminus \Omega \end{cases},$$

Trong đó γ là một tham số, $\mathcal{L}_K^p$ là toán tử không địa phương với nhân kì dị *K*, Ω là tập mở bị chặn của $\mathbb{R}^N$ với biên Lipschitz, *f* là hàm Carathéodory. Sử dụng lý thuyết Morse, chúng tôi nhận được sự tồn tại hai nghiệm của bài toán trên. Theo hiểu biết tốt nhất của chúng tôi, kết quả trong bài báo này là mới.

Từ khóa: *Toàn tử vi tích phân, toán tử p-Laplace thứ, lý thuyết Morse*

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ON EXISTENCE OF TWO NONTRIVIAL SO LUTIONS TO DIRICHLET PROBLEM INVOLVING NON-LOCAL FRACTIONAL *p*-LAPLACE

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ABSTRACT

The aim of this paper is to study the existence of solutions for Dirichlet problem involving nonlocal *p*-fractional Laplacian

$$\begin{cases} \mathcal{L}_K^p u &= \gamma f(x,u) \text{ in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases},$$

where *γ* is a parameter, $\mathcal{L}_K^p$ is a non-local operator with singular kernel *K*, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary ∂Ω, *f* is a Carathéodory function. By using Morse theory, we get the existence of two solutions of above problem. In our best knowledge, this result is new.

Keywords: *Integrodifferential operators, fractional p-Laplace equation, Morse theory.*

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1 Introduction and main result

Recently, a great attention has been focused on the study of the problem involving fractional and nonlocal operators. This type of the problem arises in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the type outcome of stochastically stabilization of Lévy processes [2, 4, 8] and reference therein. The literature on nonlocal operators and their applications is very interesting and quite large, we refer the interested reader to [1, 3, 5, 10, 16] and the references therein.

In this paper, we consdiered the existence of solution for Dirichlet problem involving fractional p-Laplace as follows:

$$\begin{cases} \mathcal{L}_K^p u & = \gamma f(x,u) \text{ in } \Omega, \\ u & = 0 \qquad\quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where γ is a parameter, $N > ps$ with $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary $\partial\Omega$, f is a Carathéodory function and $\mathcal{L}_K^p$ is a non-local operator defined as follows:

$$\mathcal{L}_K^p u(x) = 2 \int\limits_{\mathbb{R}^N} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x - y)dy$$

for $x \in \mathbb{R}^N$, and $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ is a measurable function with properties:

(K_1) $\eta K \in L^1(\mathbb{R}^N)$, where $\eta(x) = \min\{|x|^p, 1\}$;

(K_2) there exists $k_0 > 0$ such that $K(x) \geq k_0|x|^{-N-ps}$ for any $x \in \mathbb{R}^N \setminus \{0\}$;

$(K3)$ $K(x) = K(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

When $K(x) = \dfrac{1}{|x|^{N+ps}}$, the operator $\mathcal{L}_K^p$ becomes the fractional p-Laplace operator $(-\Delta)_p^s$.

In case $p = 2$, the problem (1.1) reduces to the fractional Laplace problem:

$$\begin{cases} (-\Delta)^s u & = f(x,u) \text{ in } \Omega, \\ u & = 0 \qquad\quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.2}$$

The functional framework for problem (1.2) was introduced in [11, 13]. We refer to [7, 12] for further details on the functional framework and its applications to the existence of solutions for the problem (1.2).

We give some assumptions as follows:

(f_1) $|f(x,t)| \leq a(x)|t|^q$ for all $(x,t) \in \Omega \times \mathbb{R}$, where $q \in (0,p)$ and $a > 0$, $a \in L^{\frac{p}{p-q}}(\Omega)$, and $f(x,0) = 0$.

(f_2) There exists $0 < \eta < 1$ such that $F(x,t) \geq \delta_1|t|^p$ for all $(x,t) \in \Omega \times [-\eta, \eta]$, where $\delta_1 > 0$ and $F(x,t) = \int\limits_0^t f(x,\tau)d\tau$.

Let $0 < s < 1 < p < \infty$ be real numbers and the fractional critical exponent p_s^* be defined as

$$p_s^* = \begin{cases} \dfrac{Np}{N - ps} & \text{if } sp < N \\ \infty & \text{if } sp \geq N. \end{cases}$$

Now, we recall some basic results on the spaces W and W_0. In the sequel we set $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$, where $\mathcal{O} = C\Omega \times C\Omega \subset \mathbb{R}^{2N}$.

Let W be a linear space of Lebesgue measureable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\int_Q |u(x) - u(y)|^p K(x-y)dxdy < \infty.$$

The space W is endowed with the norm defined as

$$||g||_W = ||g||_{L^p(\Omega)} + \Big(\int_Q |g(x) - g(y)|^p K(x-y)dxdy \Big)^{1/p}. \tag{1.3}$$

It is easily seen that $||.||_W$ is a norm on W (see, for instance, [16] for a proof). We shall work in the closes linear subspace

$$W_0 = \{u \in W : u(x) = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

The space W_0 is endowed with norm

$$||g||_{W_0} = \Big(\int_{\mathbb{R}^{2N}} |g(x) - g(y)|^p K(x-y)dxdy \Big)^{1/p}, \tag{1.4}$$

and $(W_0, ||.||_{W_0})$ is a uniformly convex Banach space (see [16], Lemma 2.4) and $C_0^\infty(\Omega) \subset W_0$ (see [6] and [16], Lemma 2.1).

Definition 1. We say that $u \in W_0$ is a weak solution of problem (1.1) if

$$\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dxdy = \gamma \int_{\mathbb{R}^N} f(x, u(x))\varphi(x)dx$$

for any $\varphi \in W_0$.

Theorem 2. *Assume that $(f_1), (f_2)$ hold. Then there exists $\gamma_0 > 0$ such that problem (1.1) has two nontrivial solutions in W_0 for all $\gamma \geq \gamma_0$.*

In Theorem 2, when $K(x) = \dfrac{1}{|x|^{N+ps}}$, we get immediately the result as following:

Corollary 1. *Assume that $(f_1), (f_2)$ hold. Then there exists $\gamma_0 > 0$ such that problem*

$$\begin{cases} (-\Delta)_p^s u &= \gamma f(x,u) \text{ in } \Omega, \\ u &= 0 \qquad\quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has two nontrivial solutions in W_0 for all $\gamma \geq \gamma_0$.

2 Lemma

The following result due to Xiang-Zhang-Ferrara which give the characteristic for embedding from W_0 into $L^\nu(\Omega), \nu \in [1, p_s^*]$.

Lemma 1. *[16] Let $K : \mathbb{R}^N \backslash \{0\} \to (0, +\infty)$ be a function satisfying (K_1)-$(K3)$. Then, the following assertions hold true:*
a) the embedding $W_0 \hookrightarrow L^\nu(\Omega)$ is continuous for any $\nu \in [1, p_s^*]$;
b) the embedding $W_0 \hookrightarrow L^\nu(\Omega)$ is compact for all $\nu \in [1, p_s^*)$.

From Lemma 1, we have embedding $W_0 \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous for all $\nu \in [1, p_s^*]$. Then there exists the best constant

$$S_\nu = \inf_{v \in W_0, v \neq 0} \frac{\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(x) - v(y)|^p}{(K(x-y))^{-1}} dxdy}{\Big(\int\limits_{\mathbb{R}^N} |v(x)|^\nu dx \Big)^{p/\nu}}. \tag{2.1}$$

We recall the well-know *Palais-Smale condition* (see, for instance, [14, 15] and references therein), which in our framework reads as follows:

Palais-Smale condition. Let Φ is a function in $C^1(W_0, \mathbb{R})$. The functional Φ satisfies the *Palais-Smale compactness condition* at level $c \in \mathbb{R}$ if any sequence $\{u_j\}_{j \in \mathbb{N}}$ in W_0 such that $\Phi(u_j) \to c$ and $\sup_{||\varphi||_{W_0}=1} | < \Phi'(u_j), \varphi > | \to 0$, admits a strongly convergent subsequence in W_0.

In order to study the existence of solution for problem (1.1), we consider the energy function on W_0 as follows:

$$J(u) = \frac{1}{p} \int\limits_Q |u(x) - u(y)|^p K(x-y) dxdy - \int\limits_\Omega F(x, u) dx. \tag{2.2}$$

Then from (f_1), we have $J \in C^1(W_0, \mathbb{R})$. Furthermore, we get

$$\begin{aligned} < J^{'}(u), \varphi > = & \int\limits_\Omega |u(x) - u(y)|^{p-2} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x-y) dxdy \\ & - \gamma \int\limits_\Omega f(x, u(x)) \varphi(x) dx \end{aligned}$$

for all $u, \varphi \in W_0$. Certainly, the solution of problem (1.1) is a critical point of the energy function J.

Let E be a real Banach space, let $\phi \in C^1(E, \mathbb{R})$ and let $K = \{u \in E : \phi'(u) = 0\}$. Then, the ith critical group of ϕ at an isolated critical point $u \in K$ with $\phi(u) = c$ is defined by

$$C_i(\phi, u) := H_i(\phi^c \cap U, \phi^c \cap U \setminus \{u\}),$$

$i \in \mathbb{N} := \{0, 1, 2, \dots\}$, where $\phi^c = \{u \in E : \phi(u) \leq c\}$, U is neighborhood of u, containing the unique critical point and H_* is the singular relative homology with coefficient in an Abelian group G.

We say that $u \in E$ is a homological nontrivial critical of ϕ if at least one of its critical groups is nontrivial.

Lemma 2. *[9] Assume that ϕ has a critical point $u = 0$ with $\phi(0) = 0$. Suppose that ϕ has a local linking at 0 with respect to $E = V \bigoplus W$, $k = dimV < \infty$, that is, there exists $\rho > 0$ small such that*
(i) $\phi(u) \leq 0,\ u \in V,\ ||u|| \leq \rho$;
(ii) $\phi(u) > 0,\ u \in W, ||u|| \leq \rho$.
Then $C_k(\phi, 0) \not\equiv 0$, hence 0 is a homological nontrivial critical point of ϕ.

Lemma 3. *[9] Let E be a real Banach space and let $\phi \in C^1(E, \mathbb{R})$ satisfies the (PS) condition and be bounded from below. If ϕ has a critical point that is homological nontrivial and is not a minimizer of ϕ, then ϕ has at least three critical points.*

3 Proof of Theorem 2

We know that $C_0^\infty(\Omega)$ is a dense subspace of W_0 [6]. Since $C_0^\infty(\Omega)$ is a separable space, then W_0 is also separable space. Furthermore, W_0 is a reflexive space. Then there exist $\{e_i\}_{i=1}^\infty \subset W$ and $\{e_i^*\}_{i=1}^\infty \subset W_0^*$ such that

$$W_0 = \overline{\text{span}\{e_i : i = 1, 2, \dots\}}$$

and

$$W_0{}^* = \overline{\text{span}\{e_i^* : i = 1, 2, \dots\}},$$

where $e_i^*(e_j) = \delta_{ij}$. For any $k \in \mathbb{N}$, we put

$$Y_k := \text{span}\{e_1, \dots, e_k\}$$

and

$$Z_k := \overline{\text{span}\{e_k, e_{k+1}, \dots\}}.$$

Lemma 4. *Let $1 \le q < p_s^*$ and ρ is small, for any $k \in \mathbb{N}$, let*

$$\beta_{k+1} := \sup\{||u||_{L^q(\Omega)} : u \in Z_{k+1}, ||u||_{W_0} \le \rho\}.$$

Then $\lim_{k\to\infty} \beta_{k+1} = 0$.

Proof. Indeed, suppose that this is not true, then there exist and $\varepsilon_0 > 0$ and $\{u_i\} \subset W_0$ with u_i is in Z_{k_i+1} such that $||u_i|| = 1$, $||u_i||_{L^q(\Omega)} \ge \varepsilon_0$, where $k_i \to \infty$ as $i \to \infty$. For any $v^* \in W_0^*$, there exists $w_i^* \in Y_{k_i}^*$ such that $w_i^* \to v^*$ as $i \to \infty$. Hence,

$$|v^*(u_i)| = |(v^* - w_i^*)(u_i)| \le ||u_i||_{W_0} ||v^* - w_i^*||_{W_0^*} \to 0$$

as $i \to \infty$. Then $u_i \rightharpoonup 0$ weakly in W_0. By Lemma 1, we get $u_i \to 0$ in $L^q(\Omega)$, which contradicts with $||u_i||_{L^q(\Omega)} \ge \varepsilon_0 > 0$ for all i. Thus, we must have $\beta_{k+1} \to 0$ as $k \to \infty$. □

Proof. From (f_2) and apply Lemma 2 for $E = W_0$ and $\phi = J$, $V = Y_k, W = Z_{k+1}$, Then $W_0 = Y_k \bigoplus Z_{k+1}$. We have

$$J(u) = \frac{1}{p}||u||_{W_0}^p - \gamma \int\limits_\Omega F(x, u)dx \le \frac{1}{p}||u||_{W_0}^p - \gamma\delta_1 \int\limits_\Omega |u|^p dx. \tag{3.1}$$

for $u \in Y_k$. Since Y_k is finite-dimensional, all norms on Y_k are equivalent. Therefore, there exist two positive constants $C_{k,q}$ and $\widetilde{C}_{k,q}$, depending on k, q, such that for any $u \in Y_k$

$$C_{k,q}||u||_{W_0} \le ||u||_{L^q(\Omega)} \le \widetilde{C}_{k,q}||u||_{W_0} \tag{3.2}$$

for any $q \in [1, p_s^*]$. From (3.2), we have

$$\int\limits_\Omega |u|^p dx \ge C_{k,p}^p ||u||_{W_0}^p. \tag{3.3}$$

Combine (3.1) and (3.3), there exist $\gamma_0 = \dfrac{1}{p\delta_1 C_{k,p}^p}$ such that

$$J(u) \le \Big(\frac{1}{p} - \gamma\delta_1 C_{k,p}^p\Big)||u||_{W_0}^p \le 0$$

for all $u \in Y_k, ||u||_{W_0} \le \eta$ and $\gamma \in [\gamma_0, +\infty)$.

From (f_1), we have

$$J(u) \ge \frac{1}{p}||u||_{W_0}^p - \gamma \int_\Omega a(x)|u|^q dx. \tag{3.4}$$

Using Hölder inequality and (2.1), we get

$$\begin{aligned}\int_\Omega a(x)|u|^q dx &\le \Big(\int_\Omega (a(x))^{\frac{p}{p-q}} dx\Big)^{\frac{p-q}{p}} \Big(\int_\Omega |u|^p dx\Big)^{q/p} \\ &= ||a||_{L^{\frac{p}{p-q}}(\Omega)} ||u||_{L^p(\Omega)}^q \le ||a||_{L^{\frac{2}{2-q}}(\Omega)} S_p^{-q/p}||u||_{W_0}^q. \end{aligned}\tag{3.5}$$

From (3.4) and (3.5), we get

$$J(u) \ge \frac{1}{p}||u||_{W_0}^p - \gamma||a||_{L^{\frac{p}{p-q}}(\Omega)} S_p^{-q/p}||u||_{W_0}^q. \tag{3.6}$$

Then, we get $\lim_{||u||_{W_0}\to+\infty} J(u) = +\infty$ since $q \in (0, p)$. Therefore, J is coercive. It implies J is bounded below.

From (3.5) and note that $\dfrac{\rho u}{||u||_{W_0}}$ has norm ρ for all $u \in Z_{k+1}, 0 < \rho \le \eta$, we have

$$\begin{aligned} J(u) &\ge \frac{1}{p}||u||_{W_0}^p - \gamma||a||_{L^{\frac{p}{p-q}}(\Omega)} ||\frac{\rho u}{||u||_{W_0}}||_{L^p(\Omega)}^q \frac{||u||_{W_0}^q}{\rho^q} \\ &\ge \frac{1}{p}||u||_{W_0}^p - \gamma||a||_{L^{\frac{p}{p-q}}(\Omega)} \beta_{k+1}^q \frac{||u||_{W_0}^q}{\rho^q} \\ &= ||u||_{W_0}^q\Big(\frac{1}{p}||u||_{W_0}^{p-q} - \gamma||a||_{L^{\frac{p}{p-q}}(\Omega)} \beta_{k+1}^q\rho^{-q}\Big) \end{aligned}\tag{3.7}$$

Since $\lim_{k\to\infty}\beta_{k+1} = 0$, then when k is large enough, we get

$$\gamma||a||_{L^{\frac{p}{p-q}}(\Omega)} \beta_{k+1}^q\rho^{-q} \le \frac{1}{2p}||u||_{W_0}^{p-q},$$

thus $J(u) \ge \dfrac{1}{2p}||u||_{W_0}^p > 0$ for all $0 < ||u||_{W_0} \le \rho$. Hence J satisfies Lemma 2.

Since J is coercive, then every (PS) sequence of J is bounded. Let $\{u_n\}$ is a (PS) sequence of J. Then there exists $u \in W_0$ such that $u_n \to u$ weak in W_0. By Lemma 1, we can assume that $u_n \to u$ strong in $L^p(\Omega)$.

Now, we check J satisfy the (PS) condition. Note that

$$\Big|\int\limits_{\Omega} f(x,u_n)(u_n-u)dx\Big| \le \int\limits_{\Omega} |f(x,u_n)(u_n-u)|dx$$
$$\le ||a||_{L^{\frac{p}{p-q}}(\Omega)} ||u_n-u||^q_{L^p(\Omega)} \to 0$$

as $n\to\infty$, since $u_n \to u$ strong in $L^p(\Omega)$. Similarly, we also have

$$\int\limits_{\Omega} f(x,u)(u_n-u)dx \to 0$$

as $n\to\infty$. Thus, we get

$$\lim_{n\to\infty}\int\limits_{\Omega}(f(x,u_n)-f(x,u))(u_n-u)dx = 0. \tag{3.8}$$

For each $\varphi \in W_0$, we denote B_φ the linear functional on W_0 as follows

$$B_\varphi(v) = \int\limits_{Q} |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))(v(x)-v(y))K(x-y)dxdy.$$

Clearly, by Hölder inequality, B_φ is a continuous linearly mapping on W_0 and

$$|B_\varphi(v)| \le ||\varphi||^{p-1}_{W_0}||v||_{W_0} \text{ for all } v\in W_0.$$

Obiviously, $< J'(u_j)-J'(u), u_j-u >\to 0$ since $u_j\to u$ weak in W_0 and $J'(u_j)\to 0$. Therefore, we get

$$o(1) =< J'(u_j)-J'(u), u_j-u >= (B_{u_j}(u_j-u) - B_u(u_j-u))$$
$$-\gamma\int\limits_{\Omega}(f(x,u_j)-f(x,u))(u_j-u)dx = B_{u_j}(u_j-u)-B_u(u_j-u)+o(1). \tag{3.9}$$

It is well-know that the Simion inequalities

$$|\xi-\nu|^p \le c_p(|\xi|^{p-2}\xi - |\nu|^{p-2}\nu)(\xi-\nu), \text{ for } p\ge 2,$$
$$|\xi-\nu|^p \le C_p[(|\xi|^{p-2}\xi-|\nu|^{p-2}\nu)(\xi-\nu)]^{p/2}(|\xi|^p+|\nu|^p)^{\frac{2-p}{2}} \text{ for } 1<p<2$$

and for all $\xi,\nu\in\mathbb{R}^N$, where c_p, C_p are positive constants depending only on p. Using Simion inequality, we get

$$\int\limits_{Q} |u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))(u_j(x)-u(x)-u_j(y)+u(y))K(x-y)dxdy \ge 0.$$

From (3.9) and (3.8), we have

$$\int\limits_{Q} |u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))(u_j(x)-u(x)-u_j(y)+u(y))K(x-y)dxdy \to 0$$

as $j \to \infty$. Thus, $||u_j - u||_{W_0} \to 0$. Hence $u_j \to u$ strong in W_0. Therefore, J satisfies the (PS) condition.

Combine Lemma 2 and Lemma 3, we obtain J has two nontrivial criticals which are solutions of problem (1.1). □

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